SUBHARMONIC RESONANCE IN A SYSTEM WITH A RANDOMLY VARYING NATURAL FREQUENCY

PMM Vol. 31, No. 4, 1967, pp. 747-748

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(*Received November 12, 1966*)

We shall investigate the effect of steady-state random variations in natural frequency on the amplitude of subharmonic oscillations of a system. The example considered is that of a system describable by the Duffing equation with a periodic right-hand side. It will be shown that, as in the case of oscillations near fundamental resonance [1], the indicated effect is reducible "on the average" to variations of the "equivalent" values of the natural frequencyand damping coefficient. In our case this results in a change in the domain of existence of subharmonic oscillations.

Let us investigate steady subharmonic oscillations of order 1/3 in a system described by the differential Eq.

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \Omega^2[1 + \mu\xi(t)]x + \eta x^3 = a\cos\omega_0 t \qquad (\omega_0 \approx 3\Omega) \tag{1}$$

Here a, α, Ω, ω_0, μ and η are constants and $\xi(t)$ is a steady-state centered random function. We introduce the new variable

$$y = x - b_1\cos\omega_0 t, \qquad b_1 = a(\Omega^2 - \omega_0^2)^{-1} \tag{2}$$

The process $y(t)$ corresponds (by definition) to subharmonic oscillations. Let us consider the quantities α, μ, η, and $|\omega_0 - 3\Omega|$ small and introduce the notation

$$\alpha = \varepsilon\alpha_1, \quad \mu = \sqrt{\varepsilon}\mu_1, \quad \eta = \varepsilon\eta_1, \quad \Omega^2 - ({}^1/_3\,\omega_0)^2 = \varepsilon\Omega\Delta \tag{3}$$

Let us substitute Expressions (2) and (3) into initial Expression (1). Let us then convert to the new variables z, $\bar z$ defined by the relations

$$y = {}^1/_2\,(ze^{{}^1/_3 i\omega_0 t} + \bar z e^{-{}^1/_3 i\omega_0 t}), \qquad dy/dt = {}^1/_6\, i\omega_0(ze^{{}^1/_3 i\omega_0 t} - \bar z e^{-{}^1/_3 i\omega_0 t})$$

in order to obtain the following system of equations in standard form:

$$dz/dt = \sqrt{\varepsilon}Z_1(z, \bar z, t) + \varepsilon Z_2(z, \bar z, t), \quad d\bar z/dt = \sqrt{\varepsilon}\,\bar Z_1(z, \bar z, t) + \varepsilon\bar Z_2(z, \bar z, t) \tag{4}$$

Here

$$Z_1(z, \bar z, t) = -{}^3/_2\,(i\omega_0)^{-1}\mu_1\Omega^2\xi(t)\,[z + \bar z e^{-{}^2/_3 i\omega_0 t} + b_1(e^{{}^2/_3 i\omega_0 t} + e^{-{}^4/_3 i\omega_0 t})]$$

$$\begin{aligned} Z_2(z, \bar z, t) = {} & -({}^1/_3\, i\omega_0)^{-1}\{{}^1/_2\,\Omega\Delta(z + \bar z e^{-{}^2/_3 i\omega_0 t}) + {}^1/_3\, i\omega_0\alpha_1(z - \bar z e^{-{}^2/_3 i\omega_0 t}) + \\ & + \alpha_1\omega_0 b_1(e^{{}^2/_3 i\omega_0 t} - e^{-{}^4/_3 i\omega_0 t}) + \eta_1[{}^1/_8\,(z^3 e^{{}^2/_3 i\omega_0 t} + 3z^2\bar z + 3z\bar z^2 e^{-{}^2/_3 i\omega_0 t} + \bar z^3 e^{-{}^4/_3 i\omega_0 t}) + \\ & + {}^3/_8\, b_1(z^2 e^{{}^4/_3 i\omega_0 t} + 2z\bar z e^{{}^2/_3 i\omega_0 t} + \bar z^2 + z^2 e^{-{}^2/_3 i\omega_0 t} + 2z\bar z e^{-{}^4/_3 i\omega_0 t} + \bar z^2 e^{-2i\omega_0 t}) + \\ & + {}^3/_8\, b_1^2(ze^{2i\omega_0 t} + 2z + ze^{-2i\omega_0 t} + \bar z e^{{}^4/_3 i\omega_0 t} + 2\bar z e^{-{}^2/_3 i\omega_0 t} + \bar z e^{-{}^8/_3 i\omega_0 t}) + \\ & + {}^1/_8\, b_1^3(e^{{}^8/_3 i\omega_0 t} + 3e^{{}^2/_3 i\omega_0 t} + 3e^{-{}^4/_3 i\omega_0 t} + e^{-{}^{10}/_3 i\omega_0 t})]\} \end{aligned} \tag{5}$$

(The bars denote complex conjugates).

System (4) can be investigated by means of the averaging method which involves subjecting the equations in standard form to the Krylov-Bogoliubov technique [2] and then averaging over a set of realizations. We note that the convergence of the averaging method for $\varepsilon \to 0$ is proved rigorously for problems of the above type in [3], although the method had been used previously [4] (the domain of convergence of the method was estimated in [4] on the basis of qualitative considerations). In [1] this estimate was improved somewhat by comparing the results for a linear system with results obtained by applying the method of successive substitutions to the corresponding integral equation.

On applying the above method to system (4), we obtain equations for the slowly varying components z_0, $\bar{z}_0$ of the average values of the random functions z, $\bar{z}$,

$$dz_0 / dt = \varepsilon Z_0 (z_0, \bar{z}_0), \qquad d\bar{z}_0 / dt = \varepsilon \bar{Z}_0 (z_0, \bar{z}_0) \tag{6}$$

$$Z_0 = P z_0 - {}^9/_8 (i\omega_0)^{-1} \eta_1 b_1 \bar{z}_0^2, \qquad \bar{Z}_0 = \bar{P} \bar{z}_0 + {}^9/_8 (i\omega_0)^{-1} \eta_1 b_1 z_0^2$$

$$P, \bar{P} = -\alpha_1 - {}^9/_4 \pi\mu^2\Omega^4\omega_0^{-2} [\Phi(0) - \Phi({}^2/_3\omega_0)] \pm ({}^1/_3 i\omega_0)^{-1} [{}^1/_{18}\omega_0^2 - {}^1/_2 \Omega^2 + {}^9/_4 \pi\mu^2 \Omega^4 \omega_0^{-1} \Psi({}^2/_3\omega_0) - {}^3/_8 \eta_1 b^2 - {}^3/_4 \eta_1 b_1^2]$$

$$b = \sqrt{z_0 \bar{z}_0}, \qquad \Phi(\omega) = \frac{1}{\pi}\int_0^\infty K(\tau)\cos\omega\tau\, d\tau, \qquad \Psi(\omega) = \frac{1}{\pi}\int_0^\infty K(\tau)\sin\omega\tau\, d\tau \tag{7}$$

Here $K(\tau)$ is the correlation function of the random process $\xi(t)$.

Setting the right-hand sides of Eqs. (6) equal to zero and returning to the initial notation, we find that the average amplitude b of the steady subharmonic oscillations is determined by the roots of the biquadratic Eq.,

$$({}^3/_8\eta)^2 b^4 + b^2\{{}^3/_8 \eta [\Omega_*^2 - ({}^1/_3\omega_0)^2 + {}^3/_2 \eta b_1^2] - ({}^3/_8 \eta b_1)^2\} + {}^1/_2 [\Omega_*^2 - ({}^1/_3\omega_0)^2 + {}^3/_2 \eta b_1^2] + ({}^1/_3\omega_0\alpha_*)^2 = 0 \tag{8}$$

$$\alpha_* = \alpha + {}^9/_4 \pi\mu^2\Omega^4\omega_0^{-2} [\Phi(0) - \Phi({}^2/_3\omega_0)], \quad \Omega_*^2 = \Omega^2 - {}^3/_2 \pi\mu^2\Omega^4\omega_0^{-1} \Psi({}^2/_3\omega_0) \tag{9}$$

Eq. (8) is similar in form to the familiar [5] analogous equation for a system with constant ($\mu = 0$) parameters. The effect of random variations in natural frequency thus reduces to changes in the equivalent parameters α and Ω in accordance with Formulas (9). By determining the discriminant of Eq. (8), we can find the consequent change in the boundary of the existence domain of the subharmonic oscillations.

For example, let

$$K(\tau) = \exp(-|\tau| / \tau_0), \qquad \pi\Psi({}^2/_3\omega_0) = {}^2/_3\omega_0\tau_0^2 [1 + ({}^2/_3\omega_0\tau_0)^2]^{-1}$$

$$\pi[\Phi(0) - \Phi({}^2/_3\omega_0)] = {}^4/_9 \omega_0^2\tau_0^3 [1 + ({}^2/_3\omega_0\tau_0)^2]^{-1} = {}^2/_3\omega_0\tau_0 \pi \Psi({}^2/_3\omega_0)$$

The dominant role in the case $2/3\,\omega_0\tau_0 \gg 1$ is then played by the increase in the equivalent damping coefficient. It is easy to show, in particular, that slow (as compared with the doubled period of small free oscillations) random variations in natural frequency must reduce the domain of existence of the subharmonic oscillations.

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Translated by A.Y.